International Journal of Theoretical Physics, Vol. 24, No. 2, 1985

Scalar Charged Particle in the Lorentz Gauge

Egon Marx[1]

Received March 23, 1984

Relativistic quantum mechanics leads to the specification of initial and final conditions for the wave amplitudes and electromagnetic potentials. The interaction between one scalar charged particle and the electromagnetic field has previously been solved by perturbation expansions in the Coulomb gauge. Here the theory is extended to the Lorentz gauge, which requires a different set of initial or final conditions on the potentials.

1. INTRODUCTION

Relativistic quantum mechanics offers an alternative to quantum field theory that is not beset by problems with divergent terms in perturbation expansions. The wave function can be interpreted in terms of probability amplitudes properly normalized, and pair creation and annihilation can be represented by wave functions with a fixed number of variables. This formalism is based on Stueckelberg and Feynman's idea (Stueckelberg, 1941; Feynman, 1949) of particles scattered backward in time, Dirac's many-times formalism (Dirac, 1932), and the separation of the wave function into positive- and negative-frequency parts (Feshbach and Villars, 1958). We further developed the relativistic theory of charged particles interacting with an external electromagnetic field (Marx, 1969, 1970a, 1970b).

The interaction of one scalar charged particle with the electromagnetic field was solved by means of perturbation expansions (Marx, 1979) in the Coulomb gauge. In this gauge, the gauge-dependent parts of the potentials vanish (Marx, 1970c), only the transverse part of the vector potential obeys the wave equation, and there are no problems with the initial or final conditions. The Coulomb gauge condition is not invariant under Lorentz transformations, while the Lorentz condition is invariant. The specification of the potentials at finite times requires a careful consideration of the

[1]National Bureau of Standards, Gaithersburg, Md. 20899.

0020-7748/85/0200-0217$04.50/0

constraints that result from Maxwell's equations and the Lorentz condition when the charge density does not vanish. We show here how to obtain the solution in perturbation series of the particle wave function and the electromagnetic potentials.

We repeat only those equations from Marx (1979) that are required to understand the discussion of the initial conditions. We use the same notation and natural units.

2. EQUATIONS OF MOTION AND CONSTRAINTS

The charged scalar particle is represented by the complex wave function $\phi(x)$, which satisfies the Klein–Gordon equation

$$(D^2+m^2)\phi(x)=0 \tag{1}$$

where

$$D_\mu=\partial_\mu+ieA_\mu(x) \tag{2}$$

The electromagnetic potentials A_μ are related to the fields $F_{\mu\nu}$ by

$$F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu} \tag{3}$$

they obey the Maxwell equations

$$F_{\mu\nu,\nu}=j_\mu \tag{4}$$

where j_μ stands for the charge and current densities

$$j_\mu=ie[\phi^*D_\mu\phi-(D_\mu^*\phi^*)\phi] \tag{5}$$

In a Lorentz gauge, the potentials satisfy the Lorentz condition

$$A_{\mu,\mu}=0 \tag{6}$$

and equation (4) reduces to the d'Alembert wave equation

$$\partial^2A_\mu=j_\mu \tag{7}$$

Equation (4) also implies that the charge has to be conserved, that is,

$$j_{\mu,\mu}=0 \tag{8}$$

which is satisfied as a consequence of the Klein–Gordon equation (1). We then take the four-divergence of both sides of equation (7) and find that

$$\partial^2A_{\mu,\mu}=0 \tag{9}$$

thus, if $A_{\mu,\mu}$ and $\partial_0A_{\mu,\mu}$ vanish at the initial (or final) time, the Lorentz condition is satisfied at all times. Also one of the Maxwell equations (4), which can be rewritten as

$$\nabla\cdot\mathbf{E}=j_0 \tag{10}$$

is a constraint that has to be satisfied only at the initial (or final) time. In terms of the potentials, this equation is

$$\nabla^2 A_0 + \nabla \cdot \dot{\mathbf{A}} = -j_0 \tag{11}$$

The wave function ϕ can be separated into a positive-frequency part $\phi^{(+)}$ and a negative-frequency part $\phi^{(-)}$, which are related by a simple operator to the probability amplitudes for the particle and the antiparticle, respectively. The boundary conditions in relativistic quantum mechanics are such that we have to specify the particle amplitude at the initial time and the antiparticle amplitude at the final time. To describe particle scattering combined with pair annihilation, we specify the particle amplitude at the initial time and set the antiparticle amplitude at the final time equal to zero; through the equations of motion, we then find the solution in the form of the positive-frequency part at the final time, which provides the probability amplitude for particle scattering, and the negative-frequency part at the initial time, which provides the probability amplitude for pair annihilation. The wave function can also be obtained at intermediate times, but we do not relate this quantity to physical measurements. Similarly, we specify the antiparticle amplitude at the final time and set the particle amplitude at the initial time equal to zero to describe antiparticle scattering and pair creation. These boundary conditions arise naturally if we use the causal Green's function or Feynman propagator to solve the Klein–Gordon equation.

The electromagnetic potential is a real field, so we should not use the (complex) causal Green's function for the d'Alembert equation. Furthermore, specification of either the positive- or negative-frequency part of a real field requires that both the function and the time derivative be given, but the potentials would be overspecified if we give them and their time derivatives both at the initial and at the final times. Thus, we specify the electromagnetic potentials completely either at the initial time (if the particle amplitude is given) or at the final time (if the antiparticle amplitude is given) and use the retarded or advanced Green's function, respectively, to solve the equation. There still is a degree of arbitrariness in the boundary values of the potentials owing to gauge invariance; we specify a free electromagnetic field by giving two solenoidal vector fields for the values of the vector potential and its time derivative at the boundary time.

3. PERTURBATION EXPANSIONS IN THE LORENTZ GAUGE

In this section we discuss the problem of particle scattering and pair annihilation; the solution of antiparticle scattering plus pair creation is the same *mutatis mutandis*.

We assume that we can expand the fields in a power series in the charge e, that is, we set

$$\phi(x)=\sum_{k=0}^{\infty} e^k\phi^{(k)}(x) \tag{12}$$

$$A_\mu(x)=\sum_{k=0}^{\infty} e^k A_\mu^{(k)}(x) \tag{13}$$

and, separating the equations in each order, we obtain

$$(\partial^2+m^2)\phi^{(k)}(x)=\omega^{(k)}(x) \tag{14}$$

$$\partial^2 A_\mu^{(k)}(x)=j_\mu^{(k)}(x) \tag{15}$$

where the sources $\omega^{(k)}$ and $j_\mu^{(k)}$ are expressed in terms of lower-order contributions to the fields (Marx, 1979). The Lorentz condition and the charge conservation equation also are satisfied in each order of e.

In zeroth order, the sources vanish and the fields are determined by their initial conditions. In particular, we have $\nabla\cdot\mathbf{A}^{(0)}=0$ from the initial conditions and we set $A_0^{(0)}=0$, so that we satisfy both the conditions for the Coulomb and Lorentz gauges to this order, as well as the constraint (11).

The positive-frequency part $\phi^{(+)}$ of the wave function has to satisfy the initial condition in zeroth order, and it vanishes at the initial time t_i in higher orders. The negative-frequency part $\phi^{(-)}$ can have contributions in each higher order, and both parts affect the charge density at t_i. We have to take into account these contributions when computing the scalar potential, so that all constraints are satisfied. We have some leeway in the selection of initial conditions, because there are transformations that lead from one Lorentz gauge to another if the gauge function satisfies the d'Alembert equation. We satisfy the constraints if we set

$$\nabla^2 A_0^{(k)}(\mathbf{x},t_i)=-j_0^{(k)}(\mathbf{x},t_i) \tag{16}$$

$$\partial_0 A_0^{(k)}(\mathbf{x},t_i)=0 \tag{17}$$

$$\mathbf{A}^{(k)}(\mathbf{x},t_i)=0, \qquad k>0 \tag{18}$$

$$\partial_0\mathbf{A}^{(k)}(\mathbf{x},t_i)=0, \qquad k>0 \tag{19}$$

The solution of the Poisson equation (16),

$$A_0^{(k)}(\mathbf{x},t_i)=\frac{1}{4\pi}\int d^3x'\frac{j_0^{(k)}(\mathbf{x}',t_i)}{|\mathbf{x}-\mathbf{x}'|} \tag{20}$$

allows us to find $A_0^{(k)}$ at the initial time; we can then solve the d'Alembert equation (15) with the help of the retarded Green's function and obtain

$$A_0^{(k)}(\mathbf{x},t)=\frac{1}{4\pi}\int d^3x'|\mathbf{x}-\mathbf{x}'|^{-1}[j_0^{(k)}(\mathbf{x}',\tau)+\delta'(\tau)A_0^{(k)}(\mathbf{x}',t_i)] \tag{21}$$

where $\delta'(\tau)$ is the derivative of the Dirac δ function of the retarded time $\tau = t - t_i - |\mathbf{x} - \mathbf{x}'|$; we note that $j_0^{(k)}(\mathbf{x}, t)$ vanishes for $t < t_i$.

Thus, in order $k > 0$, the constraint equation (11) is satisfied as a consequence of equations (16) and (19), the Lorentz condition is satisfied at t_i from equations (17) and (18), and the derivative of the Lorentz condition also vanishes at t_i owing to equation (19) and

$$\partial_0^2 A_0^{(k)}(\mathbf{x}, t_i) = 0 \tag{22}$$

which is a consequence of the d'Alembert equation (15) and equation (16).

4. CONCLUDING REMARKS

We have shown how the constraints on the electromagnetic potentials can be satisfied in the Lorentz gauge when $\phi^{(+)}(\mathbf{x}, t_i)$ is given. If $\phi^{(-)}(\mathbf{x}, t_f)$ is given instead, we have to specify the potentials at the final time t_f and use the advanced Green's function to solve the d'Alembert equation.

Although this solution is more complicated than the solution in the Coulomb gauge, there are no difficulties in principle to carry out the calculations to determine the terms in the perturbation expansion in the Lorentz gauge. These solutions are not formally Lorentz covariant, which is a reflection of the fact that, although usually the equations in a relativistic theory are covariant, the boundary conditions are not.

REFERENCES

Dirac, P. A. M. (1932). *Proc. R. Soc., (London)* **136**, 453.
Feshbach, H., and Villars, F. (1958). *Rev. Mod. Phys.*, **30**, 24.
Feynman, R. P. (1949). *Phys. Rev.*, **76**, 749, 769.
Marx, E. (1969). *Nuovo Cimento*, **60A**, 669.
Marx, E. (1970a). *Nuovo Cimento*, **67A**, 129.
Marx, E. (1970b). *Int. J. Theor. Phys.*, **3**, 401.
Marx, E. (1970c). *Int. J. Theor. Phys.*, **3**, 467.
Marx, E. (1979). *Int. J. Theor. Phys.*, **18**, 819.
Stueckelberg, E. C. G. (1941). *Helv. Phys. Acta*, **14**, 588.